ESTIMATING THE YIELD CURVE FOR SOVEREIGN BONDS: THE CASE OF TURKEY

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Abstract

In this study, we estimate the yield curves for Turkish sovereign bond market for each month of the 2005-2018 period utilizing two common methods used in the literature; namely Extended Nelson-Siegel (ENS) and Dynamic Nelson-Siegel (DNS) models. Within our sample period, the Turkish sovereign bond market gradually becomes more liquid and the introduction of the 10-year fixed-rate coupon bonds after 2010 makes it possible to reliably estimate yield curves with up to 10-year maturities. Given the findings of our analyses and considering the sum of squared errors, we conclude that DNS performs slightly better than ENS in estimating the shapes of the yield curves. This can be partly due to the 'local optima' problem that is common in the non-linear optimization approach utilized in ENS methodology.

Keywords: Interest Rates, Yield Curve, Term Structure, Sovereign Bonds, Nelson-Siegel Model

JEL Classification: C53, C82, E43, G12

Özet

Bu makalede, Türkiye devlet tahvilleri piyasasının 2005-2018 arasındaki aylık verim eğrilerini literatürde en yaygın kullanılan iki metot olan Extended Nelson-Siegel (ENS) ve Dynamic Nelson-Siegel (DNS) modellerini kullanarak tahmin etmekteyiz. Örneklem dönemimiz içerisinde, devlet tahvilleri piyasası daha likit hale geldiğinden ve 10 yıl vadeye sahip kupon bonolarının ihraç edilmeye başlandığı için (2010 sonrası), daha sağlıklı verim eğrisi tahminlerini 10 yıl vadeye kadar yapmamız mümkün olmuştur. Analizlerimiz sonucunda ve karesel hatalar toplamı dikkate alındığında, verim eğrilerinin şekillerini tahmin etmede DNS modelinin ENS modeline nazaran bir miktar daha başarılı olduğu sonucuna varmaktayız. Bunun kısmen, ENS metodunda kullanılan ve doğrusal olmayan optimizasyon yaklaşımında sık rastlanan 'lokal optimum' sorunuyla ilgili olduğu söylenebilir.

Anahtar Kelimeler: Faiz Oranları, Verim Eğrisi, Vade Yapısı, Devlet Tahvilleri, Nelson-Siegel Modeli

JEL Sınıflandırması: C53, C82, E43, G12

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1. Introduction

Yield curve (also called 'spot rate curve') reflects the 'term structure' of interest rates, i.e., the relationship between the interest rates and the time to maturity of a security. The main benefit of estimating a yield curve for the bond market is extracting information about interest rates clear of the influence of fluctuations in interest rate of specific bonds. Consequently, accurate estimation of yield curves is crucial for making sound monetary policy decisions and efficient portfolio management. Since the markets do not have discount bonds for every conceivable maturity traded on a daily basis, the way to numerically estimate the yield curve is by using limited number of securities that are available in the marketplace with specific durations to maturity.

The finance literature has developed several methodologies used in yield curve estimation. Bliss and Fama (1987) use a 'smoothed bootstrap' method to extract future and spot rates from estimated data and then estimate the curve via regression. There are curve-fitting methods that include numerous estimated parameters, such as quadratic and cubic splines (McCulloch, 1971 and 1975), exponential splines (Vasicek and Fong, 1982), basis splines (Steeley, 1991), maximum smoothness splines (Adams and Deventer, 1994) and roughness penalty function splines (Fisher et al., 1994; Waggoner, 1997). Another strand of literature offers stochastic models for the short-term rates. Among these, some follow equilibrium arguments to model the dynamics of the instantaneous rate and obtain yields at other maturities under specific assumptions about risk premium. Vasicek (1977), Cox et al. (1985), Duffie and Kan (1996) are important contributors to the equilibrium approach. Some others use the no-arbitrage approach, which tries to fit the yield curve to existing instrument prices at a point in time in a way to avoid any arbitrage opportunities. Brennan and Schwartz (1979) and Ho and Lee (1986), Hull and White (1990) are some of the earliest contributors to the no-arbitrage methodology while Heath, Jarrow, Morton (1992) offer a multifactor version by modelling the entire forward curve as opposed to simple short rate. These stochastic models are mainly used for pricing financial derivatives.

Among these methods developed in the last five decades, the most popular ones are the parametric models. Among this group, Nelson and Siegel (1987) model, its extension by Svensson (1994, 1996) and its dynamic version by Diebold and Li (2006) are widely used by central banks, academia and other market participants for estimating the yield curve. Nelson and Siegel build a static model, which fits the yield curve to the observed market prices of bonds.

The purpose of this article is to estimate yield curve for Turkish government bonds and analyze the changes in their shapes over time, applying the Nelson-Siegel model's more recent variations; namely the Extended Nelson-Siegel (ENS) and Dynamic Nelson-Siegel (DNS) models. We have chosen these two methods since they are the ones most commonly used by many financial institutions and there is a consensus in literature on the quality of their fit.[4] Although both ENS and DNS estimate very similar shapes for the yield curve at different dates, when sum of squared deviations between theoretical and market prices of securities are compared, DNS, using ordinary least square (OLS) regressions, seems to be performing better in estimating yield curves compared to the non-linear optimization technique used for calibration in ENS.

The studies on Turkish yield curve modeling include Akıncı et. al (2007), Alper et. al (2007), Iren (2009), Yıldız (2017) among others. This study contributes to this literature in multiple ways. The time period included in this study is more extensive than several of the studies in the literature. The methodology mostly relies on raw bond prices in calibration rather than rough estimates of interest rates. Furthermore, the comparison of two methods provides guidance to future endeavors in modeling the interest rates in Turkey.

The paper is organized as follows: Section 2 provides a detailed explanation of the Nelson-Siegel model and its extensions. Section 3 introduces the sample of Turkish sovereign bond market and details of our data set used in estimating the yield curves. Section 4 presents the calibration results of yield curve parameters by two alternative methods. Section 5 compares their results and finally, section 6 concludes.

2. Methodology: Extended (ENS) and Dynamic (DNS) Nelson-Siegel Models

Nelson and Siegel (1987) estimate the yield curve by fitting market prices to a parametrized curve with four parameters. Accordingly, a zero-coupon discount yield *r*, for a maturity of *T* years, is a function of time and four parameters, as shown in the below equation:

$$r(T) = \beta_0 + (\beta_1 + \beta_2)\frac{1 - e^{-\frac{T}{\lambda}}}{\frac{T}{\lambda}} - \beta_2 e^{-\frac{T}{\lambda}} \qquad (2.1)$$

[4] For example see Gilli, Große and Schumann (2010), Gimenoa and Nave (2009), BIS (2005) and references therein, among others published by academics, central banks and industry groups.

The parameters β_0, β_1, β_2 and λ are calibrated either through the actual bond prices as we do here or an estimated set of bootstrapped interest rates as exemplified by Diebold and Li (2006). Nelson and Siegel (1987) interpret their parameters as follows: β_0 is the long-term component, which remains constant when term to maturity (T) evolves, β_1 represents the short-term and β_2 the medium-term component. As Ibanez (2016) states, the fourth parameter (λ), which is not fully described by Nelson and Siegel, is a decay factor; which means it influences the fit of the model over longer time to maturities.

2.1. The Extended Nelson-Siegel (ENS) Model of Svensson (1994):

This model extends the Nelson and Siegel's model by including two additional parameters to the equation above. While the main idea behind the calibration remains the same, ENS model can better capture the occasional non-linearities observed in yield curves in practice and improve the fit of the model. ENS model estimates the yield curve by using the following equation:

$$r(T) = \beta_0 + (\beta_1 + \beta_2)\frac{1 - e^{-\frac{T}{\lambda_1}}}{\frac{T}{\lambda_1}} - \beta_2 e^{-\frac{T}{\lambda_1}} + \beta_3\left(\frac{1 - e^{-\frac{T}{\lambda_2}}}{\frac{T}{\lambda_2}} - e^{-\frac{T}{\lambda_2}}\right) \qquad (2.2)$$

One can utilize the *'generalized reduced gradient (GRG) non-linear optimization method'* to minimize sum of squared errors between the market price and theoretical price and select the set of parameters offering the best fit.

2.2. The Dynamic Nelson-Siegel (DNS) Model of Diebold and Li (2006):

Diebold and Li (2006) propose a slight variation on the Nelson-Siegel model to capture the time-varying parameters in a numerically stable manner and estimate the yield curve using the following equation:

$$r(T) = \beta_1 + \beta_2\left(\frac{1 - e^{-\lambda T}}{\lambda T}\right) + \beta_3\left(\frac{1 - e^{-\lambda T}}{\lambda T} - e^{-\lambda T}\right) \qquad (2.3)$$

The most important contribution of this approach to the literature is arguably that the DNS parameters that represent the yield curve can be used for forecasting purposes as well. Diebold and Li reinterprets the parameters in this equation as level (β_1), slope (β_2) and curvature (β_3). Although the four free parameters could be estimated by non-linear least squares methods, Diebold and Li prefer to fix λ at a pre-defined value so as to increase the reliability of betas. Then, the beta parameters can be estimated by OLS because non-linearity in the equation is eliminated, after fixing λ. Therefore, if DNS model is to be calibrated via an estimate of existing bond yields, the calibration reduces to a simple linear regression.

Another advantage of DNS model is that the remaining three free parameters can be interpreted to have financial factors that can be determined from actual data. The three parameters of the model are interpreted such that the level parameter (β_1) represents the long-term factor (i.e., $\beta_1 = r(\infty)$); the slope parameter (β_2) represents the relative difference between long-term and short-term interest rates; and the curvature parameter (β_3) represents the infliction point of the curve, meaning the medium term that interest rate function changes in convexity.

These model-implied factors are closely related to their counterparts estimated directly from interest rate data. We have to modify Diebold and Li (2006)'s definitions slightly to fit the case of Turkish financial markets, where the time horizon for 'long-term' is noticeably different than that of the more mature US market. Accordingly, we estimate the level parameter (β_1) from data via $r(60)$ meaning, with a slight abuse of notation in changing T from years to months, the interest rate corresponding to 60 months or 5 years as the long-term factor. The slope parameter (β_2), is similarly extracted from data via $r(60) – r(3)$, the difference between 5-year and 3-month rates and the curvature parameter via $2r(24) - (r(60) + r(3))$. Diebold and Li (2006) claim that the model implied parameters resulting from regression model and the parameters implied by the interest rate data used as input should be close as a sign of their model's estimation power.

In the succeeding sections, we will estimate yield curves for the Turkish sovereign bond market using these two methodologies, namely ENS and DNS.

3. Sample

Our sample consists of end-of-month observations (last business day of every month) on Turkish sovereign bonds and bills in the period between February 2005 and December 2018. The prolonged time span of our data set is one of the contributions of this study, as the existing term structure studies on Turkey focus on much shorter time ranges. Furthermore, the Turkish sovereign debt market has evolved over the last two decades to become more liquid and started to offer instruments with gradually longer maturities, which allows us to quantify yield curves with longer maturities.

The starting date of February 2005 is chosen because it is the date that fixed-rate coupon bonds with as long as 5-year maturities started to be traded in Turkey. In this period, Turkish financial markets have also become more transparent and accountable, hence more trustworthy, with the introduction of new regulations following the devastating financial crisis in early 2000s. Moreover, 10-year fixed-rate coupon bonds started to be traded only after January 2010. Hence, while 10-year yield curves are estimated for the 2010-2018 period, yield curves with only 5-year term to maturity are estimated for the period of 2005-2010. We believe that estimating a 10-year yield curve without having a security with 10-year maturity would give incorrect/imprecise estimation results.

The data is retrieved from the web site of Borsa İstanbul, BIST (formerly İstanbul Stock Exchange)[5]. Each day's data reports valuation date, days to maturity, days to coupon, accrued interest, price(s), simple and compound rate of return and transaction volume of each bond. Our sample consists observations for 167 months; henceforth we estimate 167 separate yield curves, one for each month[6]. Although both fixed- and floating-rate coupon bonds are issued in this period, we only use Turkish Lira denominated zero-coupon and fixed-coupon rate bonds in our yield curve estimations since cash flows of floating rate bonds cannot be determined in advance.

[5] Debt Securities Market Daily Bulletin reports are posted at the website: www.borsaistanbul.com

[6] All results are available upon request.

Table 1: Descriptive Statistics of Instruments for the Period of Study (2005-2018)

Averages	All Instruments	Included	Excluded
Number of Instruments	20,5	8,2	12,3
Weighted Price	95,42	96,80	94,50
Compounded Yield	12,35	12,06	12,54
Time to Maturity (days)	919	1.097	800
Trade Volume (TL)	805.178.827	546.270.359	258.908.467

Since we believe that the prices of illiquid bonds could skew the estimation results and estimated yield curves would not reflect the reality, in order to eliminate the bias in estimations due to those relatively illiquid securities, we limited our calibration study to include only the most liquid bonds with maturities covering the range of maturities we are interested in. Accordingly, we have selected between 7 to 9 most heavily traded bonds (in volume) each day, with maturity dates within the intervals of 0-3 months, 3-6 months, 6-12 months, 1-2 years, 1-3 years, 3-5 years, 5-7 years and 7-10 years. For instance, if the last business day of the month includes data about 30 sovereign bonds, we have chosen only the 9 most liquid bonds, one for each interval. The descriptive statistics reported in Table 1 yield a detailed picture of the instruments considered and show that the selected sample used in our analyses is highly successful in representing the overall conditions in Turkish bond market within the sampling period.

Since Borsa İstanbul's daily bulletin data does not include coupon rates of the bonds, we calculated them by using accrued interest and with the following formula[7]:

$$Coupon\ rate = \frac{364 \times \text{Accrued Interest}}{182 - \text{Days to Coupon}} \tag{3.1}$$

Weighted average price is used as clean price and dirty price is calculated by summing up quoted clean price and accrued interest of each bond. We convert the discretely compounded yields to continuously compounded ones to match the approach of Nelson-Siegel models, as both of the models try to capture the dynamics of continuously compounded

[7] The chosen Turkish sovereign bonds pay coupon once every 182 days (semi-annually), whereas twice as many days makes up a year, per the day-count convention for Turkish sovereign bonds.

rates. The theoretical price can be computed via standard bond pricing techniques for a given set of parameters of the ENS model used to obtain interest rates. For each of the models, ENS and DNS, we can calculate the zero-coupon rate *r(T)* for each maturity *T* by using formulas (2.2) or (2.3), respectively. With the interest rates at hand, we can compute the present values of each coupon and principal payment. Hence, we reach the parameters calibrating the model to match the actual market prices of the bonds in our sample.

An important issue to note here is that Diebold and Li use Fama-Bliss interest rates bootstrapped from bond prices as inputs in their regression analyses. Borsa Istanbul bulletin, on the other hand, reports simple yields to maturity. However, Demirel (2014) reports that the differences between bootstrapped zero-coupon yields and the reported bond yields are very small. In our un-reported estimations, we tested this claim as well, for select dates and bonds and concurred with the result reported in Demirel (2014). Thus, we directly use the reported yields as inputs in our OLS estimations used in DNS model. Since the reported yields have irregular maturities, it is somewhat difficult to provide a descriptive picture of inputs. However, the average maturity and yield reported in Table 1 should give the reader an idea about the inputs.

4. Estimation Strategy and Results

We can obtain ENS and DNS theoretical prices by summing up the present values of all coupon and principal payments for each individual bond. In order to calibrate the model parameters to match the market prices, we choose to minimize the sum of squared deviations between the theoretical price and the market price, for all the *N* bonds chosen for each date, as shown in equation (4.1) below:

$$\min_{(\beta_0,\beta_1,\beta_2,\beta_3,\lambda_1,\lambda_2)} \sum_{n=1}^{N} \left(P_i^{ENS/DNS\ Model\ Price} - P_i^{Dirty\ Market\ Price} \right)^2 \tag{4.1}$$

This optimization problem can be solved by GRG (generalized reduced gradient) non-linear optimization method, a pre-programmed method available in most statistics software packages[8], in order to minimize the sum of the squared deviations between dirty price and

[8] We ran our analyses on both MS Excel and Matlab. Although embedded optimization methods were not exactly the same, we achieved highly similar results. The reported optimization results are based on Excel output.

theoretical prices for the ENS model. For the DNS model, we choose to follow Diebold and Li (2006) and apply ordinary least squares. Diebold and Li (2006) interpret the parameter λ as the maturity at which the loading on the curvature factor (β_3) reaches its maximum and fix λ at a pre-defined value by assuming 30 months as the medium term for the US sovereign bond market. Since Diebold and Li works with the bond market of a specific developed economy, instead of using the value they calculate for λ, we wanted to fix λ at a value so as to better reflect Turkish bond market. Duran (2014), for example, fixes λ at 1.017 in estimating Turkey yield curve between 2010 and 2014.

For our sample, we find that the loading on the medium-term (curvature) factor is maximized at around 24 months. Hence, the curvature factor (β_3) reaches its maximum at λ=0.897 when 24 months is assumed as the medium term for the Turkish sovereign bond market. Hence, after deciding to fix λ at 0.897, we used ordinary least squares regressions for estimating betas, per the DNS method used by Diebold and Li. However, we do not fix λ values for non-linear optimization techniques used for ENS estimations. The two values used in the ENS model are attributed to the first and second curvature measures for the yield curve. One can pick fixed values for both to improve the convergence of the model. In our case, we forgo this option and treat these two λ values as free parameters. The result is a better fit for the curves at the expense of accuracy (i.e., higher variance) of the parameters, as we keep a total of six free parameters.

4.1. Extended Nelson–Siegel Model and Non-Linear Optimization

This section reports the results of ENS model for the 167 months included in our study period. The range of maturities go from 1 month to 5 years prior to 2010, and up to 10 years after 2010[9]. In Table 2, we report the descriptive statistics of model parameters. Table 3 presents estimates of interest rates for selected maturities and the subsequent figures provide a visual representation of the estimated monthly yield curves.

Due to limited space, we cannot report estimation results separately for each yield curve estimated for each date in our sample, but to give the readers an idea about how the estimated shapes of those 167 yield curves evolve over time; we present them together in Graph 1 with the descriptive statistics of the ENS interest rate estimates reported in Table 3 below.

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9 As explained above, this is due to the fact that Turkish government began issuing bonds with 10-year maturities only after 2010.

We want to note that the study period covers financially tumultuous period in the aftermath of 2008 Global Financial Crisis, as well as locally-induced volatile periods observed in 2013 and most recently in summer 2018. The spikes in the level of interest rates and relative change in the slopes attest to the reflection of those periods in the term structure dynamics of interest rates. In fact, as can be surmised from the descriptive statistics, the average yield curve during that period is inverted, with long-term rates lower than short-term rates. This inverted yield curve is usually interpreted as a sign of economic distress. Furthermore, the model also captures the volatility of short term rates, which is expected to be higher than longer maturity rates.

Table 2: Descriptive Statistics of Extended Nelson-Siegel Model Parameters

Parameter	*Beta 0*	*Beta 1*	*Beta 2*	*Beta 3*	*Lambda 1*	*Lambda 2*
Mean	0.1220	-2.0226	-2.3405	80.0859	69.8935	39370.6440
Standard Error	0.0035	1.1286	1.8295	414.6260	14.4738	4890.4504
Median	0.1056	0.1154	0.1270	-0.0310	0.0000	11411.7641
Standard Deviation	0.0448	14.5851	23.6421	5358.1490	187.0429	63198.5471
Kurtosis	0.7319	33.1905	6.7726	9.2783	11.1110	3.1903
Skewness	0.9729	-4.4850	-0.4319	-1.7016	3.3495	1.9505
Range	0.2287	182.6218	204.4844	37281.0683	991.3849	287867.7095
Minimum	0.0451	-118.8899	-118.9370	-25356.0531	0.0000	0.0987
Maximum	0.2737	63.7320	85.5474	11925.0152	991.3849	287867.8081
Count	167	167	167	167	167	167

Although ENS method is very successful in fitting the term structure to the actual bond data, one shortcoming it has over other less sophisticated methods is that its parameters are sometimes instable as can be seen in the extreme values in Table 2. This instability of parameters, however, does not hurt the efficacy of the model in estimates. One can partially solve that problem by fixing lambda values via optimization (see Gilli et. al, 2008 as an example), but we forgo that process here as improving optimization method is beyond the scope of this study. Furthermore, its parameters does not have direct economic interpretation. We next use DNS method whose parameters have straightforward interpretations.

Table 3: Descriptive Statistics of ENS Interest Rates for Selected Maturities

Maturity (years)	0.25	0.5	1	2	5	7	10
Mean	0.1157	0.1158	0.1166	0.1176	0.1158	0.0980	0.0972
Standard Error	0.0034	0.0034	0.0034	0.0033	0.0028	0.0022	0.0018
Median	0.1004	0.1018	0.1037	0.1027	0.1023	0.0932	0.0937
Standard Deviation	0.0438	0.0439	0.0435	0.0421	0.0363	0.0231	0.0194
Kurtosis	0.2852	0.0895	-0.1229	-0.2204	-0.2428	5.5231	3.9067
Skewness	0.8973	0.8456	0.8022	0.7967	0.7969	1.9750	1.6342
Range	0.2277	0.2242	0.2174	0.2042	0.1676	0.1451	0.1132
Minimum	0.0461	0.0470	0.0488	0.0519	0.0583	0.0609	0.0630
Maximum	0.2738	0.2712	0.2662	0.2561	0.2259	0.2060	0.1762
Count	167	167	167	167	167	108	108

4.2. Dynamic Nelson–Siegel Model and Ordinary Least Squares (OLS) Regression

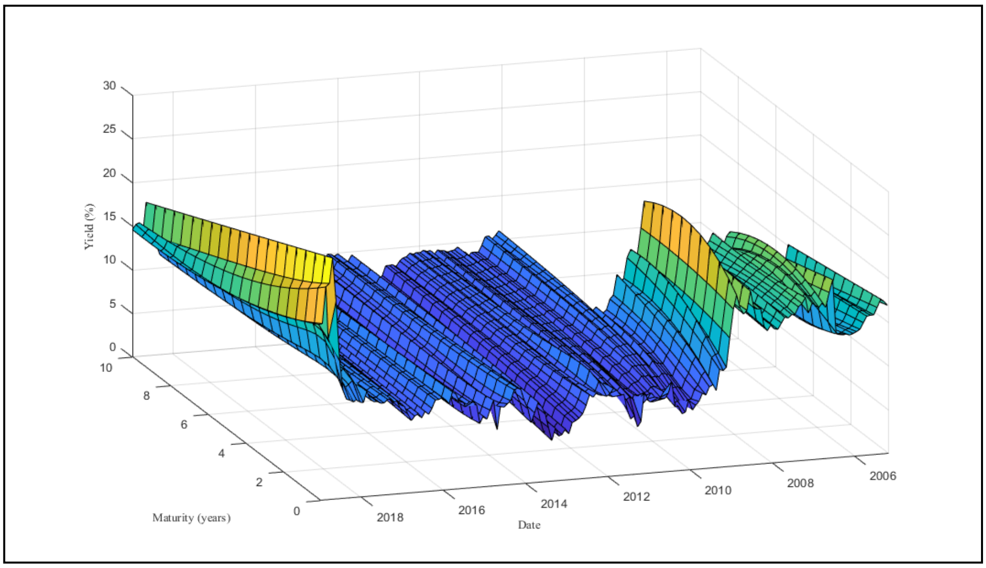
As explained above, Diebold and Li (2006) propose a dynamic approach for estimating yield curves. They use estimates of interest rates for existing maturities as inputs. By fixing λ to minimize the mean square errors in equation (2.3), they reduce the function to a linear one. Hence, one can apply ordinary least squares regression (OLS) to extract beta values (β_1, β_2 and β_3) as coefficients of a linear equation.

As we mention in the previous section, following Diebold and Li's arguments, we optimize the model over λ values to yield the minimum mean squared errors. The resulting optimal λ value comes out to be 0.897. With this result at hand, we can run OLS regressions using yield data for each month. We do not have space for all of those 167 regression results[10] for each of the yield curves for 167 months but report just the summary statistics in Table 4 below. Here, we want to emphasize that the estimated beta coefficients are mostly statistically significant[11] at conventional levels; moreover, the adjusted R^2 values are always above 90%. Descriptive statistics of DNS interest rates for selected maturities are reported in Table 5 below.

[10] All estimation results for each date are available upon request.

[11] All of the β_1 (level) coefficients were statistically significant at 10%.

Graph 1: Turkish Sovereign Yield Curves Estimated via Extended Nelson-Siegel Model



Note: The yield curves modeled with parameter calibration using non-linear optimization technique to match bond prices. The results are for 167 months between February 2005 and December 2018. Maturities up to 5 years are estimated until January 2010, after which bonds with maturities up to 10 years have become available and used in the analysis.

As can be surmised from the graphical representation of the estimated yield curves below in Graph 2, the yield curve dynamics capture the variation in interest rate dynamics relatively successfully. In fact, the inversion in yield curve and volatility difference between long and short-term rates can be seen in the selected results reported in Table 5, similar in nature to ENS results even though they slightly differ in estimates. After this relatively broad look at the DNS results, we next summarize the time series of parameter estimates.

Table 4: Descriptive Statistics of Dynamic Nelson-Siegel Model Parameters

Parameters	Beta 1 (Level)	Beta 2 (Slope)	Beta 3 (Curvature)
Mean	0.1189	-0.0033	0.0298
Standard Error	0.0028	0.0023	0.0042
Median	0.1049	-0.0010	0.0140
Standard Deviation	0.0357	0.0302	0.0548
Sample Variance	0.0013	0.0009	0.0030
Kurtosis	0.2809	2.2620	3.7397
Skewness	1.0421	0.4425	1.7176
Range	0.1701	0.2057	0.3526
Minimum	0.0673	-0.0878	-0.0568
Maximum	0.2375	0.1179	0.2957
Count	167	167	167

Table 5: Descriptive Statistics of DNS Interest Rates for Selected Maturities

Maturity (years)	0.25	0.5	1	2	5	7	10
Mean	0.1188	0.1212	0.1242	0.1263	0.1244	0.1024	0.1019
Standard Error	0.0036	0.0037	0.0038	0.0037	0.0033	0.0027	0.0025
Median	0.1048	0.1058	0.1088	0.1091	0.1081	0.0968	0.0974
Standard Deviation	0.0468	0.0480	0.0492	0.0484	0.0425	0.0278	0.0256
Kurtosis	-0.0476	-0.1936	-0.2336	-0.1352	-0.0751	8.3580	7.8920
Skewness	0.8064	0.7988	0.8156	0.8479	0.8640	2.4891	2.3919
Range	0.2261	0.2241	0.2225	0.2332	0.2006	0.1831	0.1679
Minimum	0.0485	0.0491	0.0504	0.0534	0.0596	0.0616	0.0633
Maximum	0.2747	0.2731	0.2729	0.2865	0.2601	0.2447	0.2312
Count	167	167	167	167	167	108	108

4.3. Time Series of Estimated Parameters for DNS

The estimated DNS parameters via OLS methodology can be interpreted as level, slope and curvature of the term structure of interest rates. As can be surmised from our results and explained below, that is also true for the case of Turkish economy and yield curves. The importance these parameters are further elevated through studies[12] that show the parameters of term structure can also help forecast the future macroeconomic conditions.

The parameters of DNS model, unlike parameters of ENS, do not have large variations over time. When changes in the shape of the level parameter are observed, it can be an indication of fundamental changes in financial markets, hence long-term interest rates are expected to be influenced. After the global crisis of 2008, the Turkish economy experienced a relatively more stable period, which is reflected in our results via lower estimated values for the level parameter. On the other hand, economic and politic tribulations after 2016 are reflected with higher values for the level parameter. These trends can be seen in Graph 3 below.

The slope and curvature parameters represent the short and medium-term factors, respectively. It can be seen that both have higher volatility in the pre-crisis period (2005-2009). Similarly, their volatility goes up again in 2018 when the Turkish economy was facing macroeconomic turbulence.

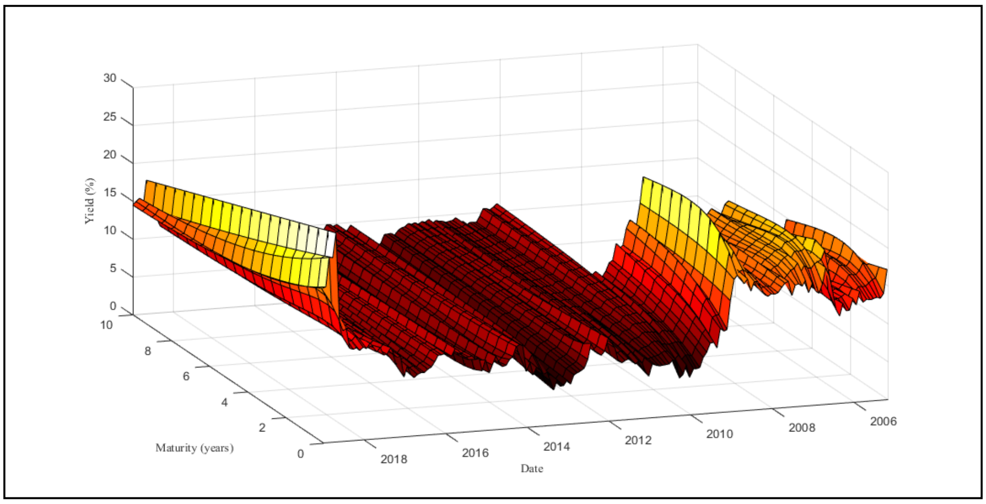
By the nature of numerical calibration, a perfect fit between the estimations and the actual market data is never possible. However, we can confidently state that both ENS and DNS models perform admirably well in modelling the term structure of interest rates for our sample, even in a market as volatile as Turkey. The discrepancy between the prices estimated by the models and the actual prices of bonds are negligibly small in most cases as can be seen in the visual documentation of sum of squared errors for each month in Graph 4. ENS calibration already yields these error terms. Comparable error terms are computed for DNS implied bond prices for comparison.

Aside from a spike in August 2018, a highly volatile time for Turkish financial markets with apparent inconsistencies in bond prices, the errors are negligibly small for both methods. Looking at the aggregate performance of the models over all 167 estimated yield curves, the sum of squared deviations is 129.2 in ENS methodology, whereas the sum is

[12] Forecasting macroeconomic factors via term structure factors is beyond the scope of this study. We refer the interested readers to the literature originally initiated by studies like Diebold, Rudebusch and Aruoba (2006) and Ang, Piazzesi and Wei (2006).

101.3 for DNS methodology. Sum of squared errors are found to be < 2 for 94% of our monthly yield curve estimations with the ENS model, while the same figure is 96% for the estimations with DNS. We can conclude that DNS estimates yield curve for Turkish bond market slightly better than ENS, when the sum of squared errors between the theoretical prices and the dirty prices of securities is considered as the success criteria.

Graph 2: Turkish Sovereign Yield Curve Estimated via Dynamic Nelson-Siegel Model



Note: The yield curve modeled with the calibration of yield curve to bond yields with ordinary least squares method. The results are for 167 months between February 2005 and December 2018. Maturities up to 5 years are estimated until 2010, after which bonds with maturities up to 10 years have become available and are included.

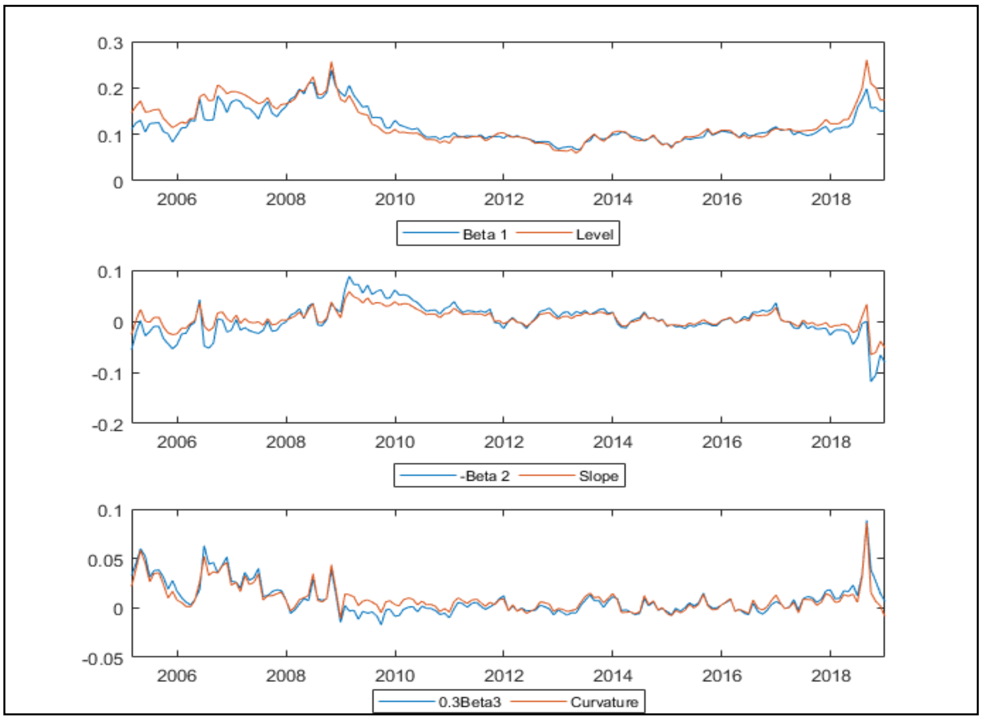
Similar to Diebold and Li's results[13] who report high correlation between level, slope, curvature parameters based on the model and corresponding parameters inferred from actual interest rate data, we also detect high correlations between these three parameters for the case of Turkey. The correlations between the estimated factors (β_1, β_2 and β_3) and the empirical level (L_t), slope (S_t) and curvature (C_t) factors, as defined above, are as calculated as follows:

[13] Diebold and Li (2006) report the following values: corr (β_1, L_t) = 0.97, corr (β_2, S_t) = -0.99 and corr (β_3, C_t) = 0.99.

$corr\ (\beta_1, L_t) = 0.93,\ corr\ (\beta_2, S_t) = -0.96,\ corr\ (\beta_3, C_t) = 0.94$

Therefore, we can accept β_1, β_2 and β_3 as three important characteristics of the estimated yield curves with β_1 representing level, - β_2 representing slope and $0.3\beta_3$ representing curvature, respectively per Diebold and Li's findings.

Graph 3: Time Series of Estimated DNS Parameters via OLS



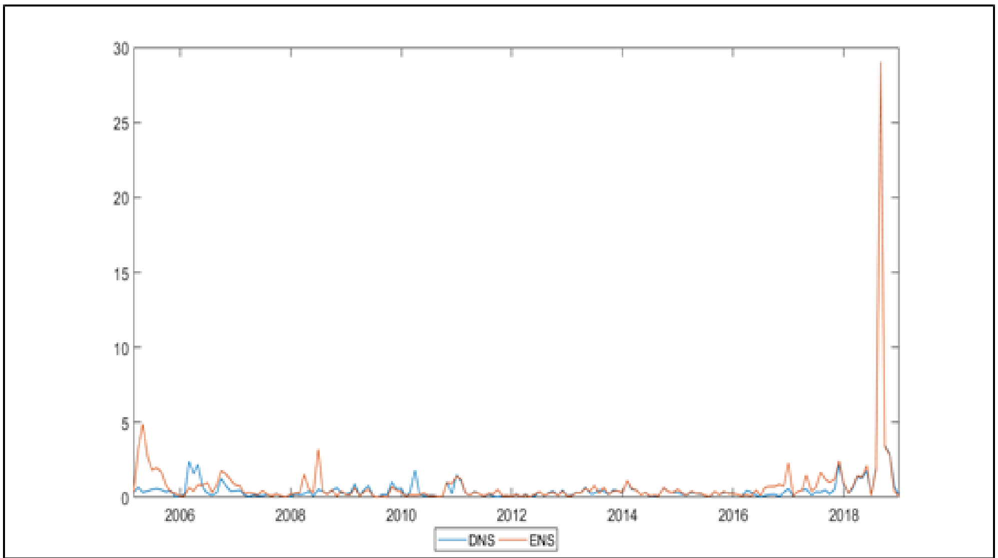
Note: Model-based level, slope and curvature (i.e., estimated factors of β_1, β_2 and β_3) and data-based level, slope and curvature. Model based parameters are the coefficients of the regression model, whereas data-based parameters are estimated from the specific interest rates that are used as input to the regression model. Refer to section 2.2 for detailed explanations.

5. Comparison of Results: ENS vs. DNS

In this section, we compare the efficacy of our estimates via each of the two methods used above. In model calibration, one can select an objective function such as minimizing mean absolute errors, mean squared errors, sum of squared errors etc. Among the possible criteria available, we chose minimizing sum of squared errors in bond prices. This criteria is in line with Nelson and Siegel (1987), Svenson (1994, 1996) and Diebold and Li (2006).

We furthermore include specific examples of fitted yields in Graph 5 for selected dates. The dates are chosen to represent a broad range of term structure dynamics, including upward and downward sloping as well as humped-shaped curves with larger curvature. According to the results, both ENS and DNS are capable of capturing different shapes of yield curves in most cases. One specific example, however, shows that they both may fail to capture the curvature of actual yields, such as August 2015. It is worth mentioning that these convergence issues can be somewhat diminished by using a weighting scheme to put more weight on short-term data points (inverse of duration can be used) or using zero-coupon yields as inputs. However, this does not take away from the relative success of each method.

Graph 4: Time Series of Sum of Squared Errors in Bond Prices for ENS and DNS



Note: Comparison of the two methods in sum of squared errors between model-implied bond prices and actual market bond prices for each calibration period.

It is also evident that ENS is more successful especially for humped-shaped curves (such as the yield curve on 30.03.2012, which can be seen in the upper-right panel of Graph 5) with more pronounced curvature. This result is expected because ENS model has been developed as an extension with a higher number of free parameters and can capture highly non-linear dynamics in yield curves.

Nonetheless, minimizing the difference between theoretical price and dirty price via non-linear optimization technique yields imperfect estimates of the yield curves. According to literature, this problem stems from the problem of 'local minima'[14]. Our non-linear estimation used to achieve ENS results, which occasionally underestimates the actual yields with a certain level of error, suffers from the same shortcoming of every available quantitative method used in non-linear calibration. We can estimate the yield curve via ENS relatively well, though far from ideal. One can improve the fit with more sophisticated optimization methods. However, this is beyond the scope of this study.

All in all, if we compare the two methodologies, namely ENS with non-linear optimization and DNS with OLS regression, we conclude that each has its own advantages and disadvantages. Non-linear optimization captures the shape of the yield curve better, yet can suffer from non-convergence to global minima in the calibration phase. OLS can fit the yields better with its ease of convergence and intuitively interpreted parameters, yet may not be able capture the extreme curvature in some cases.

6. Conclusion

Shapes of the yield curves, which show the variation in interest rates over different maturities, provide useful signals with potential important implications on the investment preferences of firms, borrowing strategies of treasuries, risk management practices of banks and monetary policy decisions of central banks. Therefore, accurately estimating the shape (hence, the parameters) of the yield curve is highly essential. This is why there are so many attempts and methodologies in literature, which try to reach the most accurate estimation results.

In this paper, we estimate yield curves for sovereign bonds traded in Turkish financial markets, utilizing two of the most renowned parametric approaches, namely Extended Nelson-Siegel (ENS) and Dynamic Nelson-Siegel (DNS) methods. We use monthly data between 2005 and 2018 in order to estimate separate yield curves for each of the 167 different

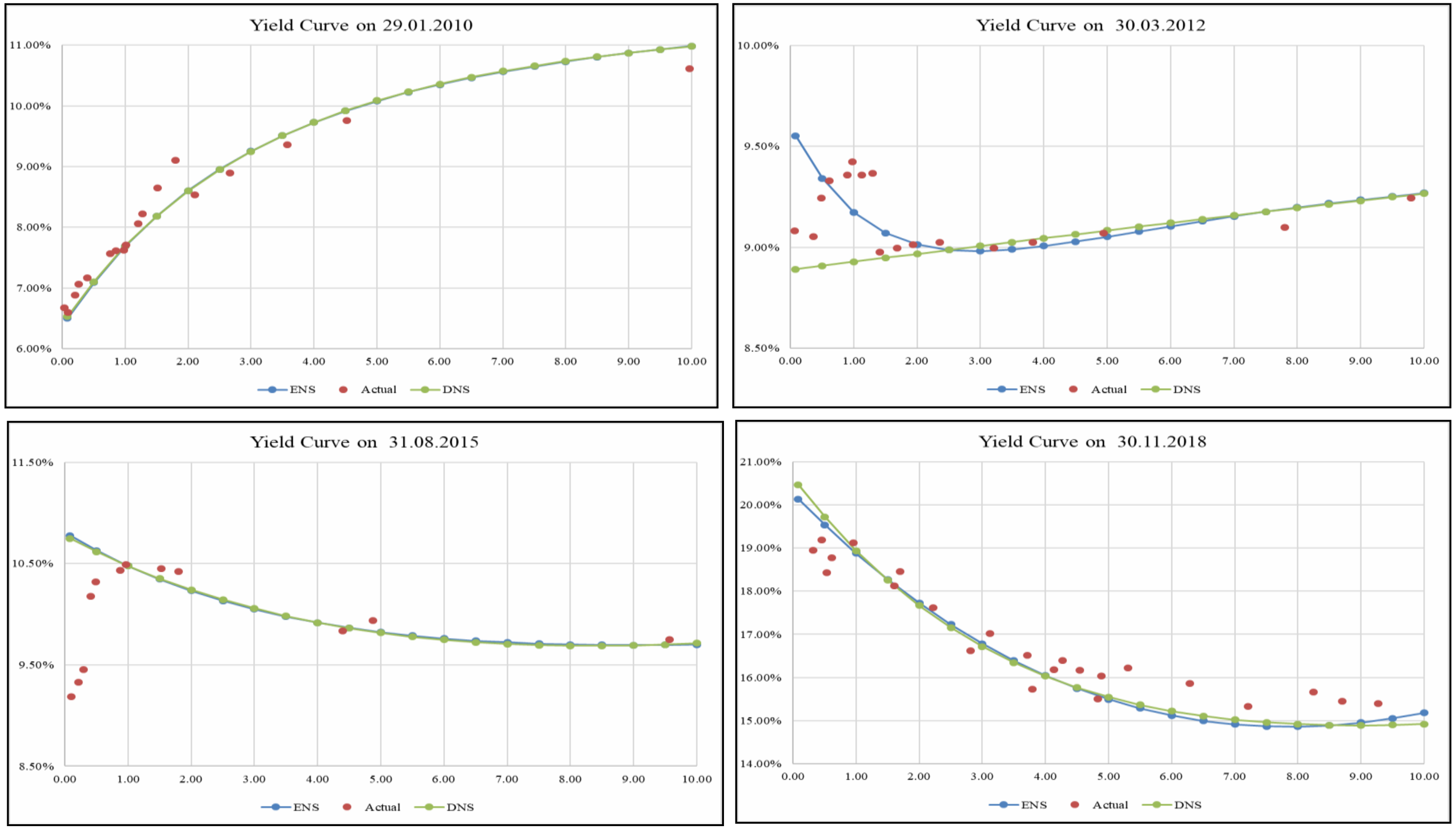
[14] Studies like Gilli et. al. (2010), Hladikova and Radova (2012) argue that the iterative optimization methods can stuck with a local minima as opposed to reaching the global minima, hence the relatively low performance of non-linear methods in convergence.

dates included in our sampling period. Estimated yield curves are up to 5-year term to maturity for the period of 2005-2010 and up to 10-year term to maturity for the 2010-2018 period, since 10-year fixed-rate coupon bonds were started to be traded only after January 2010.

Our initial expectation was that the more sophisticated ENS method, with its six parameters, to perform better in fitting yield curves; especially those with shapes that are more complicated with larger curvature and humps. However, although ENS and DNS estimations indicate highly similar trends the yield curves at different dates, the sum of squares of the differences between theoretical and dirty prices is slightly lower in the results obtained by DNS compared to those obtained by ENS. In parallel with the literature, we noticed that non-linear iterative optimization technique, unfortunately, has the potential to suffer from 'local minima' problem.

Our practice with Diebold and Li's (2006) linear approach using simple OLS regressions yield equally, if not more, accurate results with the added benefit of providing parameters with straightforward economic meaning. In line with lower values in our estimations for the level parameter, Turkish economy experienced a relatively more stable period after the global crisis of 2008. Whereas, higher values are estimated for the level parameter for the period of economic and politic tribulations observed after 2016. Similarly, the slope and curvature parameters found to have higher volatility both in the pre-global-crisis period (2005-2009) and then again in 2018 when the Turkish economy was facing macroeconomic turbulence. Hence, we can conclude that, DNS approach can be easily used not only in estimating the parameters and the shape of the yield curve, but also in forecasting future macroeconomic conditions in Turkey, which is a future research interest for the authors.

Graph 5: Sample Yield Curve Fits on Selected Dates



Note: Comparison of ENS and DNS fits the actual yield curve at various dates chosen to display yield curves with shapes of upward sloping, downward sloping curved and humped. The actual yields have been converted to continuously compounded yields to match the Nelson-Siegel outputs.

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